Brenier-Schrödinger problem - A probabilistic approach of fluid mechanics

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IRMAR

Contents

1. **From fluid evolution to Brenier-Schrödinger**
2. Girsanov and kinetic energy
3. Kinematic results
4. Existence of solutions
5. Perspectives

Euler equations (XVIII$^{\text{ème}}$) :

$$\begin{cases} \partial_t v + \nabla_v v + \nabla p = 0, \\ \operatorname{div}(v) = 0, \\ v(0,\cdot) = v_0, \end{cases}$$

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- velocity field $\to$ flow $v(t, q_t(x)) = \partial_t q_t(x)$
- incompressibility $\to$ volume preserving flow $q \in G_{\text{vol}}$
- initial data $\to$ endpoint prescription q_1

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Arnold minimisation problem ('66) :

$$\int_{[0,1]\times M} |\partial_t q_t(x)|^2 \, dt dx \to \min; \ [q_t \in G_{\text{vol}}, \ \forall 0 \le t \le 1], \ q_0 = \mathrm{id}, q_1 = h$$

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- particle evolution $\to$ distribution evolution
- paths space $\Omega = \mathcal{C}^0([0,1], \mathbb{R}^n)$, $X_t(\omega) = \omega_t$
- $Q \in \mathcal{P}(\Omega)$ path measure
- volume preserving measure $Q_t = Q(X_t \in \cdot) = \text{vol}$
- endpoint coupling $Q((X_0, X_1) \in \cdot) = \pi \in \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n)$

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Brenier's relaxation - minimisation on path measures ('89):

$$\mathbb{E}_Q \left[\int_0^1 |\dot{X}_t|^2 \, dt \right] \to \min; Q \in \mathcal{P}(\Omega), [Q_t = \mathrm{vol}, \forall 0 \le t \le 1], Q_{01} = \pi,$$

Framework :

- M compact manifold with boundary
- ν : inward-pointing normal vector field

Navier-Stokes Equations :

$$\begin{cases} \partial_t v + \nabla_v v - \Box v + \nabla p = 0, & \\ \operatorname{div}(v) = 0, & \\ \langle v, \nu \rangle = 0, & (\text{on } \partial M) \\ v(0,\cdot) = v_0. & \end{cases}$$

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Hodge - de Rahm Laplacian $\square$: suggests Brownian processes.

$$\int_0^1 |\dot{X}_t|^2 dt = +\infty \quad \text{a.s.}$$

Notion of stochastic velocity?

Brenier-Schrödinger problem

- $R \in \mathcal{P}(\Omega)$: reflected Brownian motion measure
- relative entropy

$$H(P|R) = \int_{\Omega} \log\left(\frac{dP}{dR}\right) dP$$

- $(\mu_t)_{t \in \mathcal{T}}$ in $\mathcal{P}(M)$ for $\mathcal{T} \subset [0,1]$
- $\pi \in \mathcal{P}(M^2)$

$$H(Q|R) \to \min,\ Q \in \mathcal{P}(\Omega),\ [Q_t = \mu_t,\ \forall t \in \mathcal{T}],\ Q_{01} = \pi. \qquad \text{(BS)}$$

Questions

- Link between Brenier-Schrödinger problem and kinetic energy.
- Link with Navier-Stokes equation for Nelson velocity.
- Existence of solutions.

Previous results : (Arnaudon et al. 2020) for $M = \mathbb{R}^n$ or $M = \mathbb{T}^n$.

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R refletcted Brownian motion :

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Theorem (Girsanov theorem)

Let P be such that $H(P|R) < \infty$. Then, P is the law of a semi-martingale and there exists $\zeta \in \mathcal{H}(P)$ such that P-a.s,

$$dX_t = d_m^P X_t + \zeta dt + \nu_{X_t} dL_t(X).$$

$\mathcal{H}(P)$: $g : [0,1] \times M \to TM$ such that $g_t(\omega) \in T_{\omega_t}M$, adapted and

$$\mathbb{E}_P \left[\int_0^1 |g_t|^2 dt \right] < +\infty.$$

Entropy = Stochastic kinetic energy

Föllmer formula

$$H(P|R) = H(P_0|R_0) + \frac{1}{2}\mathbb{E}_P\left[\int_0^1 |\zeta_t|^2\, dt\right]$$

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Minimal entropy = Minimal stochastic kinetic energy

Remark : Under classical Girsanov, $\int_0^1 |\zeta_t|^2\,dt < \infty$ P-a.s

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Regular solutions

$\mathcal{T} = \mathcal{T}_r \cup \mathcal{S}$
$\mathcal{T}_r$: finite union of open intervals in $[0, 1]$
$\mathcal{S}$: finite subset of $(0, 1)$ such that $\mathcal{T} \cap \mathcal{S} = \emptyset$.

Assumptions

- There exists η, p and $(\theta_s)_{s \in \mathcal{S}}$ such that

$$P = \exp\left(\eta(X_0, X_1) + \sum_{s \in \mathcal{S}} \theta_s(X_s) + \int_{\mathcal{T}_r} p_t(X_t)\, dt \right) R,$$

- "well-defined quantities and sufficient regularity".

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- "well-defined quantities and sufficient regularity".

- Formula comes from primal/dual analysis of the problem
- To a data of $\mathcal{T}$, (μ_t) and π corresponds η, (θ_s) and p.

Theorem (García-Zelada, H., 2021)

For P_1-almost all $y \in M$, the backward stochastic velocity $\overleftharpoon{v}^y$ exists and derives from a potential φ^y :

$$\overleftharpoon{v}^y_t(X) = -\nabla\varphi^y_t(X_t),\ P\text{-a.s.} \tag{1}$$

Moreover, it satisfies :

$$\begin{cases} \left(\partial_t + \nabla_{\overleftharpoon{v}^y}\right)\overleftharpoon{v}^y = \frac{1}{2}\square\overleftharpoon{v}^y - \mathbb{1}_{\mathcal{T}}(t)\nabla p, & 0 \le t < 1, t \notin \mathcal{S}, z \in M, \\ \overleftharpoon{v}^y_t - \overleftharpoon{v}^y_{t^-} = \theta_t(.), & t \in S, z \in M, \\ \langle \overleftharpoon{v}^y, \nu(z)\rangle = 0, & z \in \partial M, \\ \overleftharpoon{v}^y_0 = -\nabla\eta(., y), & z \in M. \end{cases} \tag{2}$$

- backward velocity :

$$\overleftarrow{v}_t^P = \lim_{h \to 0^+} \frac{1}{h} \mathbb{E}_P \left[\overrightarrow{X_{t-h\wedge\tau_t} X_t} | X_{[t,1]} \right]$$

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- Not satisfied by the drift (as in $\mathbb{R}^n$).
- No incompressibility condition (as in $\mathbb{R}^n$).

(ideas of proof)

Step 1. Equation for the forward potential ψ.

$$\frac{dP^x_{[0,t]}}{dR^x_{[0,t]}} = \exp\left(\sum_{s\in\mathcal{S}, s\le t} \theta_s(X_s) + \int_{\mathcal{T}\cap[0,t]} p_r(X_r)\, dr + \psi^x_t(X_t) \right),$$

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$$\frac{dP^x_{[0,t]}}{dR^x_{[0,t]}} = \exp\left(\int_0^t \langle \zeta^x_s, d^{R^x}_m X_s \rangle - \frac{1}{2} \int_0^t |\zeta^x_s|^2 \, ds \right),$$

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$$\langle \zeta^x_t, d^{R^x}_m X_t \rangle - \frac{1}{2} |\zeta^x_t|^2\, dt = \mathbb{1}_{\mathcal{S}}(t)\theta_t + p_t dt + d\psi^x_t(X_t),\ P^x\text{-a.s.}$$

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$$\overset{x\rightharpoonup}{v} = \zeta_t^x = \nabla \psi_t^x(X_t),\ dt \otimes P^x\text{-almost surely.}$$

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$$\begin{cases} \partial_t\psi_t^x + \frac{1}{2}\Delta\psi_t^x + \frac{1}{2}\|\nabla\psi_t^x\|^2 + \mathbb{1}_{\mathcal{T}}(t)p_t = 0, & t \in [0,1) \setminus \mathcal{S}, \\ \psi_t^x - \psi_{t^-}^x = -\theta_t, & t \in \mathcal{S}, \\ \langle\nabla\psi_t^x(z), \nu_z\rangle = 0, & z \in \partial M, \\ \psi_1^x = \eta(x,\cdot), & t = 1, \end{cases}$$

(ideas of proof)

Step 2. Forward velocity's "fluid equations".

$$\begin{cases} \left(\partial_t + \nabla_{\overset{x\rightharpoonup}{v}}\right) \overset{x\rightharpoonup}{v} = -\frac{1}{2}\Box \overset{x\rightharpoonup}{v} - \mathbb{1}_{\mathcal{T}}(t)\nabla p, & t \in [0,1) \setminus \mathcal{S}, \\ \overset{x\rightharpoonup}{v}_t - \overset{x\rightharpoonup}{v}_{t^-} = -\nabla \theta_t, & t \in S, \\ \langle \overset{x\rightharpoonup}{v}(z), \nu_z \rangle = 0, & z \in \partial M, \\ \overset{x\rightharpoonup}{v}_1 = \nabla\big(\eta(\cdot, x)\big), & t = 1. \end{cases}$$

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Step 3. Time reverse.

$$\overset{\leftharpoonup y}{v}_t = - \overset{y\rightharpoonup P^*}{v}_{1-t} \circ \mathrm{rev}$$

where $\mathrm{rev}(\omega)_t = \omega_{1-t}$ and $P^* = \mathrm{rev}_* P$.

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- Homogeneous spaces
- Quotient spaces

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$$H(Q|R) \to \min, \; Q \in \mathcal{P}(\Omega), \; [Q_t = \text{vol}, \forall t \in [0,1]], \; Q_{01} = \pi. \qquad \text{(iBS)}$$

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Method : If there exists a path measure Q such that $H(Q|R) < \infty$, $Q_t = \mu_t$, $\forall t \in \mathcal{T}$ and $Q_{01} = \pi$, then there exists a unique solution.

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Previous results : (Arnaudon et al., 2020) $M = \mathbb{T}^n$ with the necessary and sufficient condition $H(\pi|R_{01}) < \infty$.

(M, g) compact Riemaniann manifold. G group of isometries. $G \curvearrowright M$ transitive.

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Examples : $\mathbb{S}^n$, $\mathbb{T}^n$.

Theorem (García-Zelada, H., 2021)

(iBS) admits a unique solution if and only if $H(\pi|R_{01}) < \infty$.

(idea of the proof)

$$Q = \int_{M^3} R(\cdot | X_0 = x, X_{1/2} = z, X_1 = y)\, \sigma(dxdzdy),$$

where $\sigma(dxdzdy) = \pi(\mathrm{d}x\mathrm{d}y) vol(\mathrm{d}z)$

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- $Q_{01} = \pi$
- $H(Q|R) < \infty$. Compactness argument +

$$H(Q|R) = H(\pi|R_{01}) + \int_{M^2} H(vol|R^{xy}_{1/2})\, \pi(\mathrm{d}x\mathrm{d}y).$$

- $Q_t =$ vol. Uniqueness of isometry-invariant measure

$N = M/G$ with M Riemanian manifold and G reflections group.
$q : M \to N$, quotient, transports :

- volume measure
- Brownian motion to reflected Brownian motion
- Path measures satisfying marginals and entropy conditions

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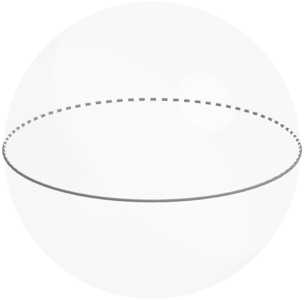
Theorem (García-Zelada, H., 2021)

If $(iBS)_{M,\pi}$ admits a solution, then $(iBS)_{N,(q\times q)_*\pi}$ admits a solution.
In particular, if M is a compact homogeneous space, then $(iBS)_{N,\tilde{\pi}}$ admits a solution for every $\tilde{\pi} \in \mathcal{P}(N^2)$ with vol_N marginals and $H(\pi|\tilde{R}_{01}) < \infty$.

Example - Disc

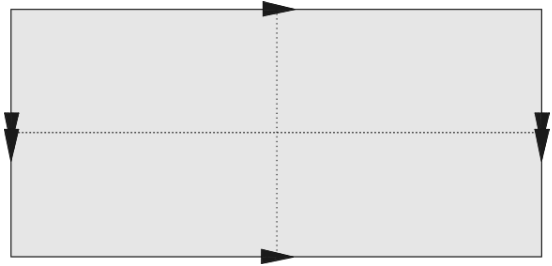
A disc (with positive curvature) can be seen as quotient space of the sphere $\mathbb{S}^2$.



$\longrightarrow$ Existence and uniqueness of solution if and only if finite entropy.

Example - Hyper-rectangle

An n-dimensional rectangular box can be seen as quotient of a torus $\mathbb{T}^n$.



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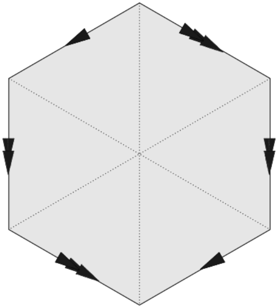
Example - Equilateral triangle

An equilateral triangle can be seen as quotient space of the torus $\mathbb{T}^2$.



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  - Hyperbolic space and its quotients (torus with 2 holes)
- Generalisation to porous media equation
  - $\partial_t v + \nabla_v v - \square v^q + \nabla p = 0$
  - Linked to Tsallis entropy $\int \frac{\mu^q - \mu}{1-q}$

Thank you for your attention.