A p-adically entire function with integral values on $\mathbb{Q}_p$ and p-adic Fourier expansions.

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We explain the magic of the entire function $\Psi_p \in \mathbb{Z}[[x]] \cap \mathbb{Q}_p\{x\}$ defined by the functional equation

$$x = \sum_{j=0}^{\infty} p^{-j} \Psi_p(p^j x)^{p^j}$$

which satisfies $\Psi_p(\mathbb{Q}_p) \subset \mathbb{Z}_p$ and admits an integral addition law. We then use the Artin-Hasse exponential

$$F(T) = \exp \sum_{i=0}^{\infty} T^{p^i}/p^i \in \mathbb{Z}_{(p)}[[T]]$$

to deduce from Ψ_p a topological basis $\{G_q\}_{q \in S}$, where $S = \mathbb{Z}[1/p] \cap \mathbb{R}_{\geq 0}$, of the Fréchet space of continuous functions $\mathbb{Q}_p \to \mathbb{Q}_p$, which consists of entire functions $G_q : \mathbb{C}_p \to \mathbb{C}_p$ defined over $\mathbb{Z}_{(p)}$ taking p-adic integral values all over $\mathbb{Q}_p$. They satisfy $G_0(x) = 1$ and

$$G_q(x+y) = \sum_{q_1+q_2=q} G_{q_1}(x) G_{q_2}(y) \text{ and } G_{pq}(px) = G_q(x)\ , \ \ \forall q \in S\ .$$

The convergence of the previous sum is uniform on compact subsets of $\mathbb{Q}_p$ along the filter of cofinite subsets of S. We identify the (p,T)-adic completion $\mathscr{D}$ of the ring $\mathbb{Z}_{(p)}[T^{1/p^\infty}]$ with a topological Hopf algebra of $\mathbb{Z}_p$-valued measures on the uniformly open subsets of $\mathbb{Q}_p$, equipped with the topology of uniform convergence on the families of balls of equal radius, in such a way that, for any $i \in \mathbb{Z}$,

$$\lim_{n \to +\infty} F(T^{p^{i-n}})^{p^n} = \Delta_{p^i}$$

is the Dirac mass at p^i. So, for the inverse series E of F, let

$$\mu_{\text{can}} := \lim_{n \to +\infty} E(\Delta_{p^{-n}} - \Delta_0)^{p^n}$$

be the $\mathbb{Z}_p$-valued measure on $\mathbb{Q}_p$ corresponding to T. Then, for any $q \in S$, the measure μ_{can}^q exists in $\mathscr{D}$ and any continuous function $f : \mathbb{Q}_p \to \mathbb{Q}_p$ admits the following generalized Amice-Fourier expansion

$$f(-) = \sum_{q \in S} \left(\int_{\mathbb{Q}_p} f \, \mu_{\text{can}}^q \right) G_q(-)\ ,$$

where the series converges as above.

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